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CONTENTS

ABSTRACT

In this paper a new equation is derived which is a quantum mechanical analog of the classical collisionless Boltzmann equation. To these ends a new quantum mechanical distribution function is employed which is particularly useful in this case. Quantum corrections for longitudinal plasma oscillations are evaluated for a low density plasma and it is found that the leading contribution ($\sim \hbar^2$) is due to exchange (Pauli-principle).

I. INTRODUCTION

A natural starting point for many investigations of quantum statistical mechanics is the use of a so-called quantum mechanical distribution function (q.m.d.f.) (Ref. 1-5). Probably the best known example of a q.m.d.f. is Wigner's function (Ref. 6) defined by

$$F(\mathbf{r}, \mathbf{k}) = (2\pi)^{-3} \int d^3y\, \rho\left(\mathbf{r} - \frac{1}{2}\,\mathbf{y}, \mathbf{r} + \frac{1}{2}\,\mathbf{y}\right) e^{i\mathbf{k}\cdot\mathbf{y}} \tag{1}$$

Here $\mathbf{k} = (m/\hbar)(\mathbf{v})$ is the wave vector and ρ is the (one particle) density-matrix. For simplicity of notation a one-particle description is used here, the generalization to N particles being straightforward. An analysis of the properties of Wigner's function will not be given since this has been done elsewhere (Ref. 7, 8). However, it is pointed out here that $F(\mathbf{r},\mathbf{k})$ as defined by Eq. (1) satisfies an equation of motion which is similar to the classical Liouville equation for the classical probability distribution in phase space. In fact, the equation satisfied by Eq. (1) goes directly over into the classical Liouville equation in the limit of $\hbar = 0$. Mean values of observables may be obtained from F in many cases by simply treating F as if it were a classical probability function. For instance, integrating Eq. (1) over all k space yields immediately the diagonal element of the density matrix which is the (observable) particle density in configuration space

$$\int F(\mathbf{r},\mathbf{k})\, d^3k = \rho(\mathbf{r},\mathbf{r}) \tag{2}$$

It is emphasized, however, that F is not an observable–which, of course, it cannot be since this would put it at variance with the exclusion principle. F is merely a calculational aid very much like the wave function.

Independently found by the author, yet already briefly mentioned in the literature (Ref. 9), is the following q.m.d.f.:

$$\tilde{F}(\mathbf{r},\mathbf{k}) = (2\pi)^{-\frac{3}{2}}\, \psi(\mathbf{r})\, c^*(\mathbf{k})\, e^{-i\,\mathbf{k}\cdot\mathbf{r}} \tag{3}$$

Here $c^*(\mathbf{k})$ is the complex conjugate of the Fourier transform of the wave function

$$c(\mathbf{k}) = (2\pi)^{-\frac{3}{2}} \int d^3r\, \psi(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

Equation (3) is valid for a pure state. For a mixed state there is

$$\tilde{F}(\mathbf{r},\mathbf{k}) = (2\pi)^{-\frac{3}{2}} \sum_n \omega_n\, \psi_n(\mathbf{r})\, c_n^*(\mathbf{k})\, e^{-i\,\mathbf{k}\cdot\mathbf{r}} \tag{5}$$

with ψ_n a complete system and ω_n the probability of occurrence of the nth state. The q.m.d.f. of Eq. (5) has already been successfully employed by the author (Ref. 10) to disentangle internal and external degrees of freedom in the Liouville equation for a gas consisting of complex molecules. It is seen in Section II that the requirements of indistinguishability of N particles may be met in a very simple way by $\tilde{F}$. This fact constitutes the real advantage $\tilde{F}$ has in this particular problem. Before proceeding to the main object of this paper, that is, the derivation of an equation for the collective motion of an electron plasma and the subsequent determination of a dispersion relation for this motion, some of the basic properties of $\tilde{F}$ will be briefly quoted.

(1) The Liouville equation satisfied by $\tilde{F}$, which is easily obtained from the definition of Eq. (3), together with the Schrodinger equation for ψ:

$$H\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = -\frac{\hbar}{i}\,\frac{\partial\psi}{\partial t} \tag{6}$$

is the following:

$$\left(\frac{\partial}{\partial t} + \frac{\hbar}{m}\,\mathbf{k}\cdot\nabla_r - i\,\frac{\hbar}{2m}\,\nabla_r^2\right)\tilde{F} = \frac{i}{\hbar}\left(e^{-i\nabla_r\cdot\nabla_k} - 1\right)V(\mathbf{r})\,\tilde{F} \tag{7}$$

The operator on the right hand side of Eq. (7) is defined by:

$$\left(e^{-i\nabla_r\cdot\nabla_k} - 1\right)V\tilde{F} = \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\,(\nabla_r\cdot\nabla_k)^n\,V(\mathbf{r})\,\tilde{F}(\mathbf{r},\mathbf{k}) \tag{8}$$

and the gradient ∇_r *only* operates on the potential $V(\mathbf{r})$. An alternative expression is possible with the aid of the identity:

$$e^{\pm\alpha\nabla_r\cdot\nabla_k}\,\phi(\mathbf{r},\mathbf{k}) = (2\pi\alpha)^{-3}\int d^3X\,d^3k'\,\phi(\mathbf{r}+\mathbf{X},\mathbf{k}')\,e^{\pm\frac{i}{\alpha}(\mathbf{k}'-\mathbf{k})\cdot\mathbf{X}} \tag{9}$$

and is

$$\left(e^{-i\nabla_r\cdot\nabla_k} - 1\right)V\tilde{F} = (2\pi)^{-3}\int d^3X\,d^3k'\,[V(\mathbf{r}+\mathbf{X}) - V(\mathbf{r})]\,\tilde{F}(\mathbf{r},\mathbf{k}')\,e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{X}} \tag{10}$$

Remembering that $\mathbf{k} = (m/h)(\mathbf{v})$, it is seen that Eq. (7) for $\tilde{F}$ goes directly over into the classical Liouville equation by taking the limit $\hbar = 0$.

$$\lim_{\hbar=0}\ \{\text{Eq. (7)}\} \rightarrow \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_r\right)\tilde{F} = \frac{1}{m}\,\nabla_r V(\mathbf{r})\cdot\nabla_v\tilde{F} \tag{11}$$

(2) The connection between $\tilde{F}$ and F already mentioned by Takabayashi (Ref. 9) is

$$e^{\frac{i}{2}\nabla_r\cdot\nabla_k}\,\tilde{F} = F \tag{12}$$

which follows directly from Eq. (3) and Eq. (9).

(3) The integral of any function of $\mathbf{r}$ and $\mathbf{k}$, $G(\mathbf{v},\mathbf{k})$, with $\tilde{F}$ over all phase space is equivalent to the quantum mechanical mean of the so-called well-ordered product (Ref. 9). In other words if

$$G(\mathbf{r},\mathbf{k}) = \sum_{\alpha,\beta} a_{\alpha\beta}\, \mathbf{r}^{\alpha} \cdot \mathbf{k}^{\beta} \tag{13}$$

then

$$\int G(\mathbf{r},\mathbf{k})\, \tilde{F}(\mathbf{r},\mathbf{k})\, d^3r\, d^3k \equiv \left\langle \psi \middle| \sum_{\alpha,\beta} a_{\alpha\beta}\, \mathbf{r}^{\alpha} \cdot \left(\frac{1}{i} \nabla_r\right)^{\beta} \middle| \psi \right\rangle \tag{14}$$

so that the momentum operator always acts on the right of the conjugate position operator.

(4) The particle density is given by:

$$\int \tilde{F}(\mathbf{r},\mathbf{k})\, d^3k = \rho(\mathbf{r},\mathbf{r}) \tag{15}$$

and

$$\int \tilde{F}(\mathbf{r},\mathbf{k})\, d^3r = \sum_{n} \omega_n \left| c_n(\mathbf{k}) \right|^2 \tag{16}$$

that is, the integration of $\tilde{F}$ over half of the phase space yields the probability distribution of the conjugate space. The analog does not always hold in more complicated cases. For instance $\hbar \int \mathbf{k} \tilde{F}\, d^3k$ is *not* the current density $\mathbf{j}$. But it is easy to show that

$$\mathbf{j} = R\,\{\hbar \int \mathbf{k} \tilde{F}(\mathbf{r},\mathbf{k})\, d^3k\} \tag{17}$$

where $R\,\{\phi\}$ means the real part of ϕ.

In Section II of this paper an equation will be derived for the singlet distribution function[1] $\tilde{F}(v,k,t)$ of an electron plasma by employing essentially the same statistical arguments of those used in the derivation of the Vlasov equation (Ref. 11). In Section III the equation obtained will be

[1] Singlet, doublet, etc., distribution functions for an N body system are here of course defined in the same way as usual (see Sect. II).

linearized. The first case then to be considered is the case of distinguishable particles. Solutions $\sim e^{i(k \cdot r - \omega t)}$ will be seen to exhibit the same dispersion relation as the "plasmons" introduced by Bohm and Pines (Ref. 12). Subsequently the dispersion relation of density fluctuations $\sim e^{i(k \cdot r - \omega t)}$ will be obtained for the case of fermions with due regard to exchange in lowest order of a perturbation expansion with respect to $\hbar \omega_p / KT$ (the ratio of a "plasma quantum" to the thermal energy, ω_p being the classical plasma frequency: $\omega_p^2 = (4\pi e^2 N)/m$). It will be shown that the leading contribution to the dispersion relation is due to exchange in this case.

II. DERIVATION OF THE QUANTUM MECHANICAL COLLISIONLESS BOLTZMANN EQUATION

The starting point of the derivation is Eq. (7) which will be written here for an N electron system:

$$\left(\frac{\partial}{\partial t} + \frac{\hbar}{m}\sum_{j=1}^{N} k_j \cdot \nabla_{r_j} - \frac{i\hbar}{2m}\sum_{j=1}^{N} \nabla^2_{r_j}\right)\tilde{F}_N(r_1 \cdots r_N, k_1 \cdots k_N, t)$$

$$= \frac{i}{\hbar}\left(e^{-i\sum_{j=1}^{N}\nabla_{r_j}\cdot\nabla_{h_j}} - 1\right)\sum_{i<j}\frac{e^2}{|r_i - r_j|}\tilde{F}_N(r_1 \cdots r_N, k_1 \cdots k_N, t) \tag{18}$$

The coulomb potential of the electrons has already been explicitly introduced. The fact that only the *electron* distribution function in Eq. (18) is considered implies that the ions are treated as an immobile background, homogeneously distributed, which does not disturb the electron distribution, with the only purpose to neutralize the space charge in thermal equilibrium. The singlet, doublet, and related distribution functions are defined as usual by:

$$\tilde{F}_S(r_1 \cdots r_S, k_1 \cdots k_S, t) = V^S \int (d^3 r\, d^3 k)^{N-S}\tilde{F}_N(r_1 \cdots r_N; k_1 \cdots k_N, t) \tag{19}$$

The integration takes place over all phase space of N-S particles as indicated by $(d^3 r\, d^3 k)^{N-S}$ in formula of Eq. (19). An equation for the singlet distribution function is obtained by integrating Eq. (18) over all sets of coordinates r_j, k_j but one. Performing this operation, observing the fact that F_n is symmetric with respect to interchange of r_i, k_i with any r_j, k_j, and discarding surface integrals in the usual fashion the following equation is obtained:

$$\left(\frac{\partial}{\partial t} + \frac{\hbar}{m}k \cdot \nabla_r - i\frac{\hbar}{2m}\nabla_r^2\right)\tilde{F}_1(r, k, t)$$

$$= \frac{ie^2 N}{\hbar}\left(e^{-i\nabla_r\cdot\nabla_k} - 1\right)\int d^3 r' d^3 k' \left|r - r'\right|^{-1}\tilde{F}_2(r, r', k, k' t) \tag{20}$$

Here N is the number of particles per cm^3. Equation (20) is not a closed equation. It entirely corresponds to the first equation of the B-B-G-K-Y hierarchy of equations in classical statistical mechanics[2]. In order to close Eq. (20) $\widetilde{F}_2$ may be expressed as a functional of $\widetilde{F}_1$. Turning for a moment to the classical case, the collisionless Boltzmann equation is obtained by merely replacing F_2 by a product of F_1 functions (see Ref. 11). This, of course, is equivalent to saying that the correlation between the particles is negligibly small. The same conclusion cannot be drawn immediately for $\widetilde{F}_2$ of Eq. (20) since it is not an observable. However, it is rather obvious that a neglect of correlation means in this case that the wave functions which determine $\widetilde{F}_2$ may be expressed by a product of properly symmetrized single particle functions in the case of indistinguishable particles. With this assmption it is easily shown that the following formula holds:

$$\widetilde{F}_2(r, r', k, k' t) = (1 + \epsilon)\widetilde{F}_1(r, k, t)\widetilde{F}_1(r', k', t) \tag{21 a}$$

with

$$\epsilon = 0 \qquad \text{for distinguishable particles}$$

$$\epsilon = \pm e^{-i(k-k')\cdot(r-r')} P_{kk'} \qquad \begin{array}{l} + \text{ sign for bosons} \\ - \text{ sign for fermions} \end{array} \tag{21 b}$$

$P_{kk'}$ is the permutation operator defined by $P_{kk'}\,\phi(k, k') = \phi(k', k)$. Equations (21a) and (21b) are proven in the appendix. A comparison of Eq. (21) with the equivalent formulation for the Wigner function in Eq. (1) as given by Ross and Kirkwood (Ref. 13) reveals the advantage of the q.m.d.f. of Eq. (5) over Eq. (1) in this case. A closed equation is now obtained from Eq. (20), using Eq. (21), for the motion of a quantum plasma which is the direct analog of the Vlasov equation (Ref. 14). It is, dropping the index 1 of the singlet distribution function:

$$\left(\frac{\partial}{\partial t} + \frac{\hbar}{m}\, k \cdot \nabla_r - i\,\frac{\hbar}{2m}\,\nabla_r^2\right)\widetilde{F}(r, k, t) = \frac{i e^2 N}{\hbar}\left(e^{-i\nabla_r\cdot\nabla_k} - 1\right) \tag{22}$$

$$\times \int d^3r'\,d^3k'\,|r - r'|^{-1}(1 + \epsilon)\,\widetilde{F}(r, k, t)\,\widetilde{F}(r', k', t)$$

---

[2] This set of equations has been derived independently by Bogolubov, Born and H. S. Green, Kirkwood and Yvon.

In order to simplify Eq. (22), the normalization from k space to velocity space is now being changed by requiring that

$$\int d^3k\,\tilde{F}(r, k, t) \equiv \int d^3v\,\tilde{F}'(r, v, t) \tag{23}$$

Dropping the prime again it is noticed that the exchange term on the right hand side of Eq. (22) may be written as (using Eq. (21b) and the substitution $m/\hbar\,(v - v') = u$)

$$\begin{aligned}\int d^3r'd^3v'\,|r_1 - r'|^{-1}\,\epsilon\tilde{F}(r, v, t)\tilde{F}(r,' v,'t) &= \\ \pm\left(\frac{\hbar}{m}\right)^3 \int d^3r'd^3u\,|r_1 - r'|^{-1}e^{-iu\cdot(r-r')} &\times \tilde{F}(r;' v - \frac{\hbar}{m}u, t)\tilde{F}(r', v, t)\end{aligned} \tag{24}$$

Here the position coordinate r is distinguished in the potential by an index since the gradient ∇_r in Eq. (22) only operates on the potential. The formula in Eq. (24) may be simplified by using the Fourier transform in velocity space for $\tilde{F}$ defined by

$$\tilde{F}(r, v, t) = (2\pi)^{-3}\int d^3l\;\tilde{f}(r, l, t)\,e^{il\cdot v} \tag{25}$$

Expressing the first d.f. on the right-hand side of Eq. (24) with the aid of Eq. (25) it is seen that

$$\begin{aligned}\int d^3r'd^3v'\,|r_1 - r|^{-1}\,\epsilon\tilde{F}(r, v, t)\tilde{F}(r,'v,'t) &= \\ \pm\left(\frac{\hbar}{m}\right)^3 \int d^3l\,|r_1 - r - \frac{\hbar}{m}l|^{-1}\tilde{f}(r, l, t)\,e^{il\cdot v} &\times \tilde{F}(r + \frac{\hbar}{m}l, v, t)\end{aligned} \tag{26}$$

Operating now with $\exp(-i\nabla_r \cdot \nabla_k) - 1$ on the expression of Eq. (26) yields, by using the formula of Eq. (9),

$$\left(e^{-i\frac{\hbar}{m}\nabla_r\cdot\nabla_v} - 1\right)\int d^3r'd^3v'\,\epsilon\tilde{F}(r, v, t)\tilde{F}(r,'v,'t) =$$

$$\pm(2\pi)^{-3}\int d^3l\,d^3X\,d^3v'\left\{\left|X - \frac{\hbar}{m}l\right|^{-1} - \frac{m}{\hbar}\,l^{-1}\right\}\tilde{f}(r, l, t) \tag{27}$$

$$\times\,\tilde{F}\left(r + \frac{\hbar}{m}l, v,'t\right)e^{-\frac{im}{\hbar}(v'-v)\cdot X + il\cdot v'}$$

In the last expression the index of r was dropped since it is not necessary any longer. The integration over X is straightforward and for the exchange part of Eq. (22) the following final expression is obtained:

$$\frac{ie^2N}{\hbar}\left(e^{-i\nabla_r\cdot\nabla_k}-1\right)\int d^3r'd^3v'\,|r-r'|^{-1}\epsilon \times \tilde{F}(r,v,t)\,\tilde{F}(r',v',t) =$$

$$\mp i\,\frac{\hbar e^2N}{m^2}\int d^3l\, e^{il\cdot v}\tilde{f}(r,l,t) \tag{28}$$

$$\times\left\{l^{-1}\tilde{F}\left(r+\frac{\hbar}{m}l, v, t\right) - \frac{1}{2\pi^2}\int d^3v'\,\frac{P}{|v-v'|^2}\,\tilde{F}\left(r+\frac{\hbar}{m}l, v', t\right)\right\}$$

where P means the principal value for the integration over v'. The $\pm$ sign refers to the two cases: bosons (+) or fermions (–). The first term on the right-hand side of Eq. (22), that is, the term without the exchange operator ϵ, can be handled in an analogous fashion. The fairly simple calculations are not reported here, but the result is quoted. The equation satisfied by $\tilde{F}(r, v, t)$ is

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla_r - i\,\frac{\hbar}{2m}\,\nabla_r^2\right)\tilde{F}(r,v,t)$$

$$= i\,\frac{e^2N}{\hbar}\int d^3r'd^3v'\,\tilde{F}(r',v',t)\,\frac{1}{2\pi^2}\int d^3u\,\frac{P}{u^2}$$

$$\times\, e^{iu\cdot(r-r')}\left[\tilde{F}\left(r, v+\frac{\hbar}{m}u, t\right)-\tilde{F}(r,v,t)\right] \mp i\,\frac{e^2N\hbar}{m^2}\int d^3l\, e^{il\cdot v'}\tilde{f}(r,l,t) \tag{29}$$

$$\times\left[l^{-1}\tilde{F}\left(r+\frac{\hbar}{m}l, v, t\right)-\frac{1}{2\pi^2}\,\frac{P}{|v-v'|^2}\,\tilde{F}\left(r+\frac{\hbar}{m}l, v', t\right)\right]$$

$\tilde{f}$ is defined by Eq. (25). Equation (29) is the starting point for the investigations of the quantum plasma. It is the quantum mechanical analog of the collisionless Boltzmann equation. Two properties of Eq. (29) are established immediately: (1) If $\tilde{F}$ is solely a function of velocity ($\tilde{F} = \tilde{F}(v)$) then Eq. (29) is identically satisfied. (2) By taking the limit $\hbar = 0$ Eq. (29) goes directly over into the classical collisionless Boltzmann equation (Vlasov equation, Ref. 14).

III. LINEARIZATION AND DETERMINATION OF DISPERSION RELATIONS

Equation (29) constitutes a rather difficult integral equation. The problems of most physical interest, however, are those in which the overwhelming majority of particles is in thermal equilibrium. Equation (29) is therefore linearized by putting

$$\tilde{F} = F_0(\mathbf{v}) + F_1(\mathbf{r}, \mathbf{v}, t) \tag{30}$$

and considering $F_1 << F_0$ so that terms quadratic in F_1 may be neglected. Specifically, it is assumed:

$$F_1(\mathbf{r}, \mathbf{v}, t) = \alpha(\mathbf{K}, \mathbf{v}, t)\, e^{i\mathbf{K}\cdot\mathbf{r}} \tag{31}$$

Entering Eq. (29) with this expression and neglecting terms of order higher than the first in α yields the following equation:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + i\mathbf{K}\cdot\mathbf{v} + i\,\frac{\hbar K^2}{2m}\right)\alpha(\mathbf{K}, \mathbf{v}, t) \\
&\quad = i\,\frac{4\pi e^2 N}{\hbar}\,\frac{1}{K^2}\int d^3v'\,\alpha(\mathbf{K}, \mathbf{v}', t)\left[F_0\left(\mathbf{v} + \frac{\hbar}{m}\mathbf{K}\right) - F_0(\mathbf{v})\right] \\
&\quad \mp i\,\frac{4\pi e^2 N\hbar}{m^2}\int d^3v'\,\frac{P}{|\mathbf{v}-\mathbf{v}'|^2}\left\{\alpha(\mathbf{K}, \mathbf{v}', t)\left[F_0(\mathbf{v}) - F_0\left(\mathbf{v} + \frac{\hbar}{m}\mathbf{K}\right)\right]\right. \\
&\quad \left. - \alpha(\mathbf{K}, \mathbf{v}, t)\left[F_0(\mathbf{v}') - F_0\left(\mathbf{v}' + \frac{\hbar}{m}\mathbf{K}\right)\right]\right\}
\end{aligned} \tag{32}$$

The last term of Eq. (32) is due to exchange. For distinguishable particles it must be omitted. Furthermore, in this case $F_0(\mathbf{v})$ is assumed to be the Maxwell-Boltzmann distribution

$$F_0(\mathbf{v}) = \left(\frac{\epsilon}{\pi}\right)^{\frac{3}{2}} e^{-\epsilon v^2} \qquad \epsilon = \frac{m}{2kT} \tag{33}$$

The equation governing the motion of the Fourier transform of Eq. (31) of the perturbed distribution, in the case in which the particles may be considered distinguishable, reads then:

$$\left(\frac{\partial}{\partial t} + i\mathbf{K}\cdot\mathbf{v} + i\,\frac{\hbar K^2}{2m}\right)\alpha(\mathbf{K},\mathbf{v},t)$$
$$= i\omega_P^2\,\frac{m}{\hbar}\,\frac{1}{K^2}\left[F_0\left(\mathbf{v}+\frac{\hbar}{m}\mathbf{K}\right) - F_0(\mathbf{v})\right]\int d^3\mathbf{v}'\alpha(\mathbf{K},\mathbf{v}',t) \tag{34}$$

The classical plasma frequency was introduced

$$\omega_P = \sqrt{\frac{4\pi e^2 N}{m}} \tag{35}$$

A solution of Eq. (34) is easily possible by employing a Laplace transformation with respect to time in exact analogy to Landau's procedure (Ref. 15). These calculations are not performed here since this would essentially mean a repetition of Landau's calculations. However, a dispersion relation for plasma oscillations is found from Eq. (34) by assuming a time dependence of α in the form

$$\alpha(\mathbf{K},\mathbf{v},t) = \beta(\mathbf{K},\mathbf{v})\,e^{-i\omega t} \tag{36}$$

Inserting Eq. (36) into Eq. (34) it is found, after division by $i(-\omega + \mathbf{K}\cdot\mathbf{v} + \hbar K^2/2m)$ and integration over $\mathbf{v}$, that:

$$1 = \omega_P^2\,\frac{m}{\hbar K^2}\int d^3v\,\frac{F_0\left(\mathbf{v}+\frac{\hbar}{m}\mathbf{K}\right) - F_0(\mathbf{v})}{-\omega + \mathbf{K}\cdot\mathbf{v} + \frac{\hbar K^2}{2m}} \tag{37}$$

which leads, after a simple substitution, to the dispersion relation

$$1 = \omega_P^2\int d^3v\,\frac{F_0(\mathbf{v})}{(\omega - \mathbf{K}\cdot\mathbf{v})^2 - \left(\frac{\hbar k^2}{2m}\right)^2} \tag{38}$$

This relation is the same as that obeyed by the "plasmons" of Bohm and Pines (Ref. 12). Two remarks of caution should be made at this point. First, it is known that α, being the Fourier transform of F_1 (Eq. 31), is not, in general, an observable. So it seems that a dispersion relation as Eq. (38) is somewhat artificial. However, Eq. (15) of the introduction shows that a simple integration over the velocity space is all that is needed to generate an observable from α, in this instance the density distribution. But this integration does not affect the spatial and temporal dependence. In other words, the (observable) density distribution displays the very same frequency versus wave-vector relationship as the (unobservable) perturbed q.m.d.f. does. Secondly, the integral in Eq. (38) is, strictly speaking, not defined because of a singularity for those values of $\mathbf{v}$ for which the denominator vanishes. This poses a problem which has been investigated by various authors for the classical case (Ref. 16). A detailed analysis cannot be entered into here but it is pointed out that the proper choice for integration in Eq. (38) is to take the principal value, and that, strictly speaking, all solutions $\sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ are damped, but this damping is particularly small if Eq. (38) is satisfied.

Now turn back to Eq. (32). The aim is to obtain the quantum corrections in lowest order to the classical dispersion relation valid for an electron plasma of low density. In order to do this, α and F_0 are expanded in powers of h by writing:

$$\left.\begin{aligned} \alpha(\mathbf{K}, \mathbf{v}, t) &= \beta(\mathbf{K}, \mathbf{v})\, e^{-i\omega t} \\ \beta &= \beta_0 + \hbar\beta_1 + \hbar^2\beta_2 + \cdots \\ \omega &= \omega_0 + \hbar\omega_1 + \hbar^2\omega_2 + \cdots \end{aligned}\right\} \tag{39}$$

The Fermi distribution is now taken for $F_0(\mathbf{v})$:

$$F_0(\mathbf{v}) = \frac{2}{N}\left(\frac{m}{\hbar}\right)^3 \left[e^{\frac{m v^2 - 2\epsilon'}{2kT}} + 1 \right]^{-1} \tag{40}$$

Since the classical limit is obtained in Ref. 17 for $\epsilon' \to \infty$, the expansion in powers of h is found correct to order $\hbar^3$:

$$
\begin{aligned}
F_0(\mathbf{v}) &= F(\mathbf{v}) + \left(\frac{2\epsilon\hbar}{m}\right)^3 \frac{N}{2} e^{-\epsilon v^2}\left(2^{-\frac{3}{2}} - e^{-\epsilon v^2}\right) \\
F(\mathbf{v}) &= \left(\frac{\epsilon}{\pi}\right)^{\frac{3}{2}} e^{-\epsilon v^2}
\end{aligned}
\tag{41}
$$

with ϵ from Eq. (33). Inserting Eq. (39) and (41) into Eq. (32) and comparing equal powers of $\hbar$ gives the following set of equations:

$$
(-\omega_0 + \mathbf{K}\cdot\mathbf{v})\beta_0 = \omega_P^2 \int d^3v'\beta_0(\mathbf{K},\mathbf{v}') \frac{\mathbf{K}\cdot\nabla_v}{K^2} F(\mathbf{v}) \tag{42}
$$

$$
\begin{aligned}
(-\omega_0 + \mathbf{K}\cdot\mathbf{v})\beta_1 + \left(-\omega_1 + \frac{K^2}{2m}\right)\beta_0 &= \omega_P^2 \int d^3v'\beta_1(\mathbf{K},\mathbf{v}') \frac{\mathbf{K}\cdot\nabla_v}{K^2} F(\mathbf{v}) \\
&+ \omega_P^2 \int d^3v'\beta_0(\mathbf{K},\mathbf{v}') \frac{(\mathbf{K}\cdot\nabla_v)^2}{2K^2 m} F(\mathbf{v})
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
-\omega_2\beta_0 &+ \left(-\omega_1 + \frac{K^2}{2m}\right)\beta_1 + (-\omega_0 + \mathbf{K}\cdot\mathbf{v})\beta_2 \\
&= \omega_P^2 \int d^3v'\beta_2(\mathbf{K},\mathbf{v}') \frac{\mathbf{K}\cdot\nabla_v}{K^2} F(\mathbf{v}) + \omega_P^2 \int d^3v'\beta_1(\mathbf{K},\mathbf{v}') \frac{(\mathbf{K}\cdot\nabla_v)^2}{2K^2 m} F(\mathbf{v}) \\
&+ \omega_P^2 \int d^3v'\beta_0(\mathbf{K},\mathbf{v}') \frac{(\mathbf{K}\cdot\nabla_v)^3}{6m^2K^2} F(\mathbf{v}) \\
&+ \frac{\omega_P^2}{m^2} \int d^3v' |\mathbf{v}-\mathbf{v}'|^{-2} \left\{\beta_0(\mathbf{K},\mathbf{v})\, \mathbf{K}\cdot\nabla_{v'} F(\mathbf{v}') - \beta_0(\mathbf{K},\mathbf{v}')\mathbf{K}\cdot\nabla_v F(\mathbf{v})\right\}
\end{aligned}
\tag{44}
$$

Equation (42), being of order $\hbar^0$, is, of course, just the classical Vlasov equation from which is obtained the well known classical dispersion relation for ω_0

$$1 = \omega_P^2 \int d^3v \, \frac{F(\mathbf{v})}{(\omega_0 - \mathbf{K} \cdot \mathbf{v})^2} \tag{45}$$

From Eq. (43) it follows, by observing Eq. (42) and (45), that

$$\omega_1 = 0 \tag{46}$$

There are no effects to first order in $\hbar$. It is a matter of simple algebra to show subsequently, by utilizing Eq. (42), (43), (45) and (46), that ω_2 from Eq. (44) is given by:

$$\omega_2 = \left[\int d^3v \, \frac{F(\mathbf{v})}{(\omega_0 - \mathbf{K} \cdot \mathbf{v})^3}\right]^{-1} \left\{\frac{1}{8} \frac{K^4}{m^2} \int d^3v \, \frac{F(\mathbf{v})}{(\omega_0 - \mathbf{K} \cdot \mathbf{v})^4} - \frac{\omega_P^2}{2m^2K^2} \int d^3v \, d^3v' \, |\mathbf{v} - \mathbf{v}'|^{-2} \, \frac{\mathbf{K} \cdot (\mathbf{v}' - \mathbf{v}) \, \mathbf{K} \cdot \nabla_{v'} F(\mathbf{v}') \, \mathbf{K} \cdot \nabla_v F(\mathbf{v})}{(\omega_0 - \mathbf{K} \cdot \mathbf{v})^2 \, (\omega_0 - \mathbf{K} \cdot \mathbf{v}')}\right\} \tag{47}$$

This, then, is the first non-vanishing quantum correction we were looking for. The expression of Eq. (47) may be simplified considerably by noting that interest lies mostly in density distributions which are spread out over regions considerably larger in volume than λ_D^3 where $\lambda_D = \sqrt{kT/4\pi e^2 N}$ is the Debye-Hückel length. In this case, an expansion in the powers of K is allowed. This expansion yields for the classical dispersion relation of Eq. (45) the well-known result

$$\omega_0 = \omega_P \left[1 + \frac{3}{2} (\lambda_D K)^2\right] \tag{48}$$

To lowest order in K Eq. (47) then leads to:

$$\omega_2 = + \frac{\omega_P}{4m^2K^2} \int d^3v \, d^3v' \, \frac{[\mathbf{K} \cdot (\mathbf{v}' - \mathbf{v})]^2}{|\mathbf{v}' - \mathbf{v}|^2} \, \mathbf{K} \cdot \nabla_v F(\mathbf{v}) \, \mathbf{K} \cdot \nabla_{v'} F(\mathbf{v}') \tag{49}$$

This integral can be done in an elementary manner with the result:

$$\omega_2 = -\frac{7}{60} \frac{\omega_P \, \epsilon}{m^2} K^2 \tag{50}$$

so that the final result is

$$\omega = \omega_0 + \hbar^2 \omega_2 = \omega_P \left\{ 1 + \left[\frac{3}{2} - \frac{7}{120} \left(\frac{\hbar \omega_P}{kT} \right)^2 \right] (\lambda_D K)^2 \right\} \tag{51}$$

The quantum correction exhibited by Eq. (51) is really a very small correction. Even when $\hbar \omega_P \approx kT$, that is, at electron densities of about 10^{15} particles per cm^3, and at room temperature, this correction is an order of magnitude smaller than the classical correction of Eq. (48).

In conclusion, it should be pointed out that Eq. (32) (the linearized collisionless Boltzmann equation for the quantum distribution function) is felt to be a good approximation also for a *high* density electron plasma—that of a metal for instance. The neglect of collisions should not be a deterrent for actual applications of Eq. (32) since the Pauli principle vastly inhibits collisions.

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APPENDIX

Here it is wished to prove the relationship of Eq. (21) which expresses the doublet distribution function by singlet functions in case of negligible interactions. The definition of Eq. (5) for $\widetilde{F}$ may be written specifically

$$\widetilde{F}_2(r, r_,'; k, k_,' t) = \sum_{\alpha,\beta} w_{\alpha\beta}\, \psi_{\alpha\beta}(r, r_;' t)\, c^*_{\alpha\beta}(k, k_,' t)\, e^{-i(k\cdot r + k'\cdot r')} \tag{A-1}$$

Here $\alpha\beta$ is a set of quantum numbers for a two-electron configuration. The singlet distribution function is given by:

$$\widetilde{F}_1(r, k, t) = \sum_{\alpha} w_\alpha\, \psi_\alpha(rt)\, c^*_\alpha(k, t)\, e^{-ik\cdot r} \tag{A-2}$$

Under the assumption of no interaction, the two-particle wave function $\psi_{\alpha\beta}$ may be expressed by

$$\psi_{\alpha\beta} = \frac{1}{\sqrt{2}} \left\{ \psi_\alpha(r)\, \psi_\beta(r') \pm \psi_\alpha(r')\, \psi_\beta(r) \right\} \tag{A-3}$$

and a similar expression for $c_{\alpha\beta}(k, k')$. Inserting (Eq. A-3) and the analogous expression for $c^*_{\alpha\beta}$ into (Eq. A-1) gives the desired result provided that

$$w_{\alpha\beta} = w_\alpha\, w_\beta \tag{A-4}$$

namely

$$\widetilde{F}_2(r, r'; k, k'; t) = \widetilde{F}_1(r, k, t)\, \widetilde{F}_1(r,'k,'t) \pm e^{-i(k-k')\cdot(r-r')} \widetilde{F}_1(r, k' t)\, \widetilde{F}_1(r,'k, t) \tag{A-5}$$

where the $\pm$ sign refers to bosons (fermions). The expression of Eq. (A-4) is of course true if particle interactions are neglected.

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